Lecture 10: Cofibration

Definition

A map $i : A \to X$ is said to have the homotopy extension property (HEP) with respect to Y if for any map $f : X \to Y$ and any homotopy $F : A \times I \to Y$ where $F(-, 0) = f \circ i$, there exists a homotopy $\bar{F} : X \times I \to Y$ such that

$$\bar{F}(i(a), t) = F(a, t), \quad F(x, 0) = f(x), \quad \forall a \in A, x \in X, t \in I.$$

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d] & X \arrow[d] \arrow[ddr, "f"] & \\
A \times I \arrow[r] \arrow[drr, "F"'] & X \times I \arrow[dr, dashed, "\bar{F}"] & \\
 & & Y .
\end{tikzcd}$$

Definition

A map $i : A \to X$ is called a cofibration if it has HEP for any spaces.

The notion of cofibration is dual to that of the fibration: fibration is defined by the HLP of the diagram

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists \tilde{F}"] & B .
\end{tikzcd}$$

If we reverse the arrows and observe that $Y \times I$ is dual to the path space Y^I via the adjointness of $(-) \times I$ and $(-)^I$, we arrive at HEP

$$\begin{tikzcd}
Y & X \arrow[l, "f"'] \arrow[dl, dashed, "\exists \tilde{F}"'] \\
Y^I \arrow[u, "p_0"] & A \arrow[l, "F"] \arrow[u, "i"'] .
\end{tikzcd}$$

Definition

Let $f\colon A \to X$. We define its mapping cylinder M_f by the push-out

$$\begin{tikzcd}
A \times \{0\} \arrow[r, hook] \arrow[d, "f"'] & A \times I \arrow[d] \arrow[ddr, "f\times 1"] & \\
X \times \{0\} \arrow[r] \arrow[drr, hook] & M_f \arrow[dr, "j"] & \\
 & & X \times I\,.
\end{tikzcd}$$

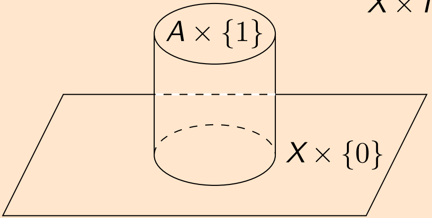


图: The mapping cylinder M_f

The mapping cylinder topology (i.e. the push-out topology) of M_f says that a map $g: M_f \to Z$ is continuous if and only if g is continuous when it is restricted to $X \times \{0\}$ and to $A \times I$.

There is a natural map $j\colon M_f \to X \times I$ induced by

$$X \times \{0\} \to X \times I, \quad f \times 1\colon A \times I \to X \times I.$$

Lemma

The HEP of $i : A \to X$ is equivalent to the property of filling the commutative diagram

$$\begin{tikzcd} M_i \arrow[r] \arrow[d] & Y \\ X \times I . \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$

Proposition

Let $i : A \to X$ and $j : M_i \to X \times I$ be defined as above. Then i is a cofibration if and only there exists $r : X \times I \to M_i$ such that $r \circ j = 1_{M_i}$.

Proposition

Let $i : A \to X$ be a cofibration. Then i is a homeomorphism to its image (i.e. embedding). If we work in $\mathscr{T}$, so A, X are compactly generated weak Hausdorff. Then i has closed image (i.e. closed inclusion).

Remark

A cofibration is not closed inclusion in general.

An example is $X = \{a, b\}$ having two points with the trivial topology and $A = \{a\}$ is one of the point.

Proof

Consider the following commutative diagram

$$\begin{tikzcd}
M_i \arrow[r, "1_{M_i}"] \arrow[d, "j"'] & M_i \\
X \times I \arrow[ur, dashed, "r"'] & .
\end{tikzcd}$$

This implies that M_i is homeomorphic to its image $j(M_i)$. Consider

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & M_i \arrow[d, "j"] \\
X \arrow[r] & X \times I .
\end{tikzcd}$$

Since $A \to M_i, M_i \to X \times I, X \to X \times I$ are all embeddings, so is $i : A \to X$.

Assume now that $A, X \in \mathscr{T}$ are compactly generated weak Hausdorff. Then the image of $j : M_i \to X \times I$ is

$$j(M_i) = (j \circ r, 1)^{-1}(\Delta_{X\times I})$$

where $\Delta_{X\times I}$ is the diagonal, hence closed subspace of $(X \times I) \times (X \times I)$. Therefore j is a closed inclusion. Since $A \to M_i, X \to X \times I$ are also closed inclusion, so is $i : A \to X$. $\square$

Proposition

Let A be a closed subspace of X. Then the inclusion map $i : A \subset X$ is a cofibration if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof.

If i is closed, then M_i is homeomorphic to the subspace $X \times \{0\} \cup A \times I$ of $X \times I$. □

Remark

If $A \subset X$ is not closed, then the mapping cylinder topology for M_i and the subspace topology for $X \times \{0\} \cup A \times I$ may not coincide.

Example

The inclusion $S^{n-1} \hookrightarrow D^n$ is a cofibration.

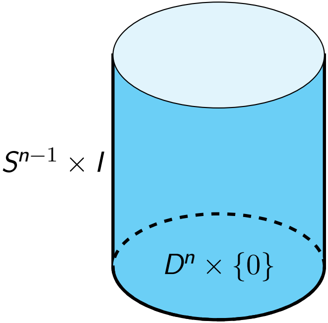


图: $D^n \times \{0\} \cup S^{n-1} \times I$ is a retract of $D^n \times I$

Proposition

Let $f\colon A \to X$ be any map. Then the closed inclusion

$$i_1 : A \to M_f, \quad a \to (a, 1)$$

is a cofibration.

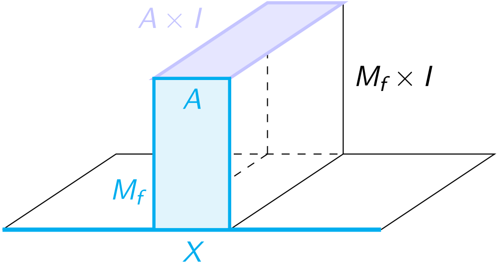


图: Retract of $M_f \times I$

Example

The inclusion $A \to A \times I, a \to a \times \{0\}$, is a cofibration.

In fact, we can view it as

$$A \to M_{1_A}$$

where $1_A : A \to A$ is the identity map.

Proposition

Let $i : A \to X$ be a cofibration, $f : A \to B$ is a map. Consider the push-out

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r] & Y . \end{tikzcd}$$

Then $j : B \to Y$ is also a cofibration. In other words, the push-out of a cofibration is a cofibration.

Proposition

Let $i : X \to Y$ and $j : Y \to Z$ be cofibrations. Then $j \circ i : X \to Z$ is also a cofibration.

Proposition

If $i : A \to X$ is a cofibration and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.

Proposition

Let $A \subset X$ and $B \subset Y$ be closed inclusions which are both cofibrations. Then the inclusion

$$X \times B \cup A \times Y \subset X \times Y$$

is also a cofibration. As a consequence, $A \times B \to X \times Y$ is a cofibration.

Let $f\colon A \to X$ be a map. Consider the diagram of mapping cylinder

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "f"'] & A \times I \arrow[d] \\
X \arrow[r] & M_f .
\end{tikzcd}$$

There is a natural commutative diagram

$$\begin{tikzcd}
& A \arrow[dl, "i_1"'] \arrow[dr, "f"] & \\
M_f \arrow[rr, "r"] & & X .
\end{tikzcd}$$

Here $i_1(a) = (a, 1), r(a, t) = f(a), r(x, 0) = x$.

Theorem

The map $r\colon M_f \to X$ is a homotopy equivalence, and $i_1 : A \to M_f$ is a cofibration. In particular, any map $f\colon A \to X$ is a composition of a cofibration with a homotopy equivalence.

This theorem says every map is equivalent to a cofibration in the homotopy category.

Definition

Let $i : A \to X, j : A \to Y$ be cofibrations. A map $f : X \to Y$ is called a cofiber map if the following diagram is commutative

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd} .$$

A cofiber homotopy between two cofiber maps $f, g : X \to Y$ is a homotopy of cofiber maps between f and g. Cofiber homotopy equivalence is defined similarly.

The following is the cofibration analogue of that for fibrations.

Proposition

Let $i : A \to X, j : A \to Y$ be cofibrations. Let $f : X \to Y$ be a cofiber map. Assume f is a homotopy equivalence. Then f is a cofiber homotopy equivalence.

Cofiber exact sequence

Now we work with the category $\underline{\mathscr{T}_\star}$ and $\underline{\mathrm{h}\mathscr{T}_\star}$. All maps and testing diagrams are required to be based.

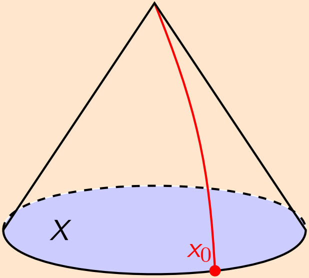
Definition

A based space (X, x_0) is called well-pointed, if the inclusion of the base point $x_0 \in X$ is a cofibration in the unbased sense.

Definition

Let $(X, x_0) \in \mathscr{T}_\star$. We define its (reduced) cone by

$$C_\star X = X \wedge I = X \times I / \left(X \times \{0\} \cup x_0 \times I \right).$$



Proposition

If X is well-pointed, then the embedding $i_1 : X \to C_\star X$ where $i_1(x) = (x, 1)$ is a cofibration.

Definition

Let $f\colon (X, x_0) \to (Y, y_0) \in \underline{\mathscr{T}_\star}$. We define its (reduced) mapping cylinder by

$$M_{\star f} = M_f/\{x_0 \times I\}.$$

If (X, x_0) is well-pointed, then the quotient $M_f \to M_{\star f}$ is a homotopy equivalence.

Definition

Given $f\colon X \to Y$ in $\mathscr{T}_\star$, we define its (reduced) homotopy cofiber $C_{\star f}$ by the push-out

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "f"'] & C_\star X \arrow[d] \\
Y \arrow[r, "j"] & C_{\star f}
\end{tikzcd}$$

If X is well-pointed, then $j : Y \to C_{\star f}$ is also a cofibration.

The quotient of $C_{\star f}$ by Y is precisely ΣX. We can extend the above maps by

$$X \longrightarrow Y \longrightarrow C_{\star f} \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C_{\star f} \longrightarrow \Sigma^2 X \longrightarrow \cdots .$$

Definition

A sequence of maps in $\underline{\mathrm{h}\mathscr{T}_\star}$

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

is called co-exact if for any $Y \in \underline{\mathrm{h}\mathscr{T}_\star}$, the following sequence of pointed sets is exact

$$\cdots \to [X_{n-1}, Y]_0 \to [X_n, Y]_0 \to [X_{n+1}, Y]_0 \to \cdots$$

Theorem (Co-exact Puppe Sequence)

Let $f\colon X \to Y$ in $\mathscr{T}_\star$ between well-pointed spaces. The following sequence is co-exact in $\mathrm{h}\mathscr{T}_\star$

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r] & C_{\star f} \arrow[r] & \Sigma X \arrow[r] & \Sigma Y \arrow[r] & \Sigma C_{\star f} \arrow[r] & \Sigma^2 X \arrow[r] & \cdots . \end{tikzcd}$$

Lemma

Let $f\colon A \to X$ be a cofibration between well-pointed spaces. Then the natural embedding

$$C_\star(A) \to C_{\star f}$$

is a cofibration.

Proof.
This follows from the push-out diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & C_\star(A) \arrow[d] \\
X \arrow[r, "j"] & C_{\star f}
\end{tikzcd}$$

□

Proposition

Let $f\colon A \to X$ be a cofibration between well-pointed spaces. Then the natural map

$$\bar{r}: C_{\star f} \to X/A$$

is a homotopy equivalence. In other words, the cofiber is homotopy equivalent to the homotopy cofiber.

Proof.

Since $C_\star(A) \to C_{\star f}$ is a cofibration and $C_\star(A)$ is constractible, the quotient

$$C_{\star f} \to C_{\star f}/C_\star(A) = X/A$$

is a homotopy equivalence. $\square$

Theorem

Let $i : A \to X$ be a cofibration between well-pointed spaces. The following sequence is co-exact in $\underline{\mathrm{h}\mathscr{T}_\star}$

$$A \longrightarrow X \longrightarrow X/A \longrightarrow \Sigma A \longrightarrow \Sigma X \longrightarrow \Sigma(X/A) \longrightarrow \Sigma^2 A \longrightarrow$$